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Inverse Heat Conduction Problem of an Elliptical Plate

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Abstract- This paper contains a heat conduction problem for an elliptical plate with heat transfer on the upper and lower surfaces, to determine the temperature with the help of Mathieu function and integral transform technique

Key Words- Inverse Thermoelastic problem, Elliptical plate, Mathieu function, Elasticity, Plane Thermal stresses, Heat transfer on plate surface.

1. INTRODUCTION

The inverse problem is very important in view of its relevance's to various industrial machines subject to heating such as main shaft of lathe and turbine, roll of a rolling mill and measurement of aerodynamic heating.

In present problem, an attempt has been made to determine the temperature distribution and unknown temperature gradient, in which the solutions are expressed in Mathieu and modified Mathieu functions with known boundary conditions using finite Marchi-Fasulo transform and Mathieu transform techniques.

2. STATEMENT OF THE PROBLEM-I

Consider on elliptic plate, and then the heat conduction equation in elliptic coordinate is given by

$$\frac{2d^{-2}}{c^2(\cosh 2\xi - \cos 2\eta)}\left[\frac{\partial^2 T}{\partial \xi^2}+\frac{\partial^2 T}{\partial \eta^2}\right]-\frac{\partial^2 (T-T_0)}{\partial z^2}=0 \qquad (1)$$

Where $c=\sqrt{a^2-b^2}$, $\xi=\tan^{-1}\left(\frac{b}{a}\right)$

Subject to interior conditions

$$T(\xi,\eta,z)=T_0 f(\eta,z) \quad a\le \xi \le b \ ,$$

$$-h\le z\le h \text{ (Known)} \qquad (2)$$

The boundary conditions are

$$\left[T(\xi,\eta,z)+k_1\frac{\partial T(\xi,\eta,z)}{\partial z}\right]_{z=h}=0 \qquad (3)$$

$$\left[T(\xi,\eta,z)+k_2\frac{\partial T(\xi,\eta,z)}{\partial z}\right]_{z=-h}=0 \qquad (4)$$

$$[T(\xi,\eta,z)]_{\xi=b}=g(\eta,z) \qquad \text{(Unknown)} \qquad (5)$$

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The equations (1) to (5) constitute the mathematical formulation of the problem under consideration.

3. SOLUTION OF THE PROBLEM

Applying finite Marchi-Fasulo transform with respect to z defined in [4] as

$$\overline{T}(\xi,\eta,\varsigma)=\int_{-h}^{h}T(\xi,\eta,z)P_m(z)dz$$

To the equations (1), (2) using (3), (4), one obtains

$$\frac{2d^{-2}}{c^2(\cosh 2\xi-\cos 2\eta)}\left[\frac{\partial^2\overline{T}}{\partial\xi^2}+\frac{\partial^2\overline{T}}{\partial\eta^2}\right]=a_m\overline{T} \qquad (6)$$

in which the Eigen values a_m are the solutions of the equation

$$[\alpha_1 a\cos(ab)+\beta_1\sin(ab)]\times[\beta_1\cos(ab)+\alpha_2 a\sin(ab)]=[\alpha_2 a\cos(ab)-\beta_2\sin(ab)]\times[\beta_1 a\cos(ab)-\alpha_1 a\sin(ab)]$$

$$\overline{T}(\xi,\eta,\varsigma)=\overline{T}_0\,\overline{f}(\eta,\varsigma) \qquad (7)$$

$$\overline{T}^*_n=f^*_n \ , \ \xi=b$$

Where $\overline{T}$ denotes the Marchi-Fasulo transform of T and ς denotes the Marchi-Fasulo transform parameter.

$$\left[\frac{\partial^2\overline{T}}{\partial\xi^2}+\frac{\partial^2\overline{T}}{\partial\eta^2}\right]=2q(\cosh 2\xi-\cos 2\eta)a_m\overline{T} \qquad (8)$$

Where $q=\frac{a_m^2 c^2}{4d^2}$

If the temperature is symmetric about both axes of elliptical plate, the appropriate solution of (8) is

$$\overline{T}(\xi,\eta,\varsigma)=\sum C_{2n}\left(f^*_n-\overline{T}^*_n\right)Ce_{2n}(\xi,-q)\times ce_{2n}(\eta,-q) \qquad (9)$$

$$\overline{f}(\xi,\eta,\varsigma)=\sum C_{2n}\left(f^*_n-\overline{T}^*_n\right)Ce_{2n}(\xi,-q)\times ce_{2n}(\eta,-q) \qquad (10)$$

$$\overline{T}_n^* = \int_0^{2\pi} \overline{T} ce_{2n}(\eta,-q)\, d\eta \tag{11}$$

$$f_n^* = \int_0^{2\pi} \overline{f}(\eta) ce_{2n}(\eta,-q)\, d\eta \tag{12}$$

In order to get the value of constant C_{2n}, multiply (10) by $Ce_{2n}(\eta, -q)$ integrate with respect to η from 0 to 2π and making use of the following result:

$$\int_0^{2\pi} Ce_{2n}^2(\eta,-q)\, d\eta = \pi \quad \text{(Orthogonal properties)}$$

$$\int_0^{2\pi} Ce_{2n}(\eta,-q)\, \overline{f}(\eta,\varsigma)\, d\eta = \overline{f}_{2n}(\varsigma), \quad \text{(Say)} \tag{13}$$

We get

$$C_{2n} = \frac{1}{\pi}\sum_{n=0}^{\infty} \frac{\overline{f}_{2n}(\xi)}{Ce_{2n}(b,-q)} \tag{14}$$

Substitute (14) in (9) one obtains

$$\overline{T}(\xi,\eta,\varsigma) = \frac{1}{\pi}\sum_{n=0}^{\infty} \frac{\overline{f}_{2n}(\varsigma)\left(f_n^* - \overline{T}_n^*\right)}{Ce_{2n}(b,-q)} Ce_{2n}(\xi,-q) \times ce_{2n}(\eta,-q) \tag{15}$$

Applying inverse Marchi-Fasulo transform to the equation (15) and using condition (5), one obtains the temperature distribution and unknown temperature gradient as

$$T(\xi,\eta,z) = \frac{1}{\pi}\sum_{m=0}^{\infty}\frac{P_m(z)}{\lambda_m}\sum_{n=0}^{\infty} \frac{\overline{f}_{2n}(\xi)\left(f_n^* - \overline{T}_n^*\right)}{Ce_{2n}(b,-q)} Ce_{2n}(\xi,-q) \times ce_{2n}(\eta,-q) \tag{16}$$

$$g(\eta,z) = \frac{1}{\pi}\sum_{m=0}^{\infty}\frac{P_m(z)}{\lambda_m}\sum_{n=0}^{\infty} \overline{f}_{2n}(\xi)\left(f_n^* - \overline{T}_n^*\right) \times ce_{2n}(\eta,-q) \tag{17}$$

where $\overline{f}_{2n}(\xi,\eta,z) = \int_{-h}^{h} f(\xi,\eta,z)\, P_m(z)\, dz$,

$$\lambda_m = \int_{-h}^{h} P_m^2(z)\, dz \ ,$$

$$P_m(z) = Q_m \cos(a_m z) - W_m \sin(a_m z)$$

$$Q_m = a_m(\alpha_1 + \alpha_2)\cos(a_m h) + (\beta_1 - \beta_2)\sin(a_m h)$$

$$W_m = (\beta_1 + \beta_2)\cos(a_m h) + (\alpha_2 - \alpha_1)\sin(a_m h)$$

Equation (16) and (17) are the desired solution of the problem with $\beta_1 = \beta_2 = 1$ and $\alpha_1 = k_1$, $\alpha_2 = k_2$.

4. STATEMENT OF THE PROBLEM-II

If we take elliptical plate with heat transfer on the upper and lower surfaces, to be two dimensional and usual elliptical coordinates (ξ,η), the heat conduction equation for solution is

$$\frac{2d^{-2}}{c^2(\cosh 2\xi - \cos 2\eta)}\left[\frac{\partial^2 T}{\partial \xi^2} + \frac{\partial^2 T}{\partial \eta^2}\right] - \frac{\partial^2 (T - T_0)}{\partial z^2} = \frac{1}{k}\frac{\partial \theta}{\partial t} \tag{18}$$

Where $c = \sqrt{a^2 - b^2}$, $\left[\xi = \tan^{-1}\left(\frac{b}{a}\right)\right]$

Subject to the initial condition

$$T(\xi,\eta,z,0) = 0 \tag{19}$$

the interior condition

$T(\xi,\eta,z,t) = T_0 f(\eta,z,t) \quad a \le \xi \le b \ , \ -h \le z \le h$ (known)

And the boundary conditions are

$$\left[T(\xi,\eta,z,t) + k_1 \frac{\partial T(\xi,\eta,z,t)}{\partial z}\right]_{z=h} = 0 \tag{20}$$

$$\left[T(\xi,\eta,z,t) + k_2 \frac{\partial T(\xi,\eta,z,t)}{\partial z}\right]_{z=-h} = 0 \tag{21}$$

$$[T(\xi,\eta,z,t)]_{\xi=b} = g(\eta,z,t) \quad \text{(Unknown)} \tag{22}$$

Where k_1 and k_2 are radiation constant on the plane surfaces. Equation (18) to (22) constitutes the mathematical formulation of the problem under consideration.

5. SOLUTION OF THE PROBLEM

Applying finite Marchi-Fasulo transform with respect to z defined in [4] as

$$\overline{T}(\xi,\eta,\varsigma,t) = \int_{-h}^{h} T(\xi,\eta,z,t) P_m(z)\, dz$$

To the equations (18), (19), (22) and using (20), (21), one obtains

$$\frac{2d^{-2}}{c^2(\cosh 2\xi - \cos 2\eta)}\left[\frac{\partial^2 \overline{T}}{\partial \xi^2} + \frac{\partial^2 \overline{T}}{\partial \eta^2}\right] = a_m \overline{T} + \frac{1}{k}\frac{\partial \overline{T}}{\partial t} \tag{23}$$

in which the Eigen values a_m are the solutions of the equation

$$[\alpha_1 a\cos(ab) + \beta_1 \sin(ab)] \times [\beta_1 \cos(ab) + \alpha_2 a \sin(ab)] = [\alpha_2 a\cos(ab) - \beta_2 \sin(ab)] \times [\beta_1 a\cos(ab) - \alpha_1 a \sin(ab)]$$

$$\overline{T}(\xi,\eta,\varsigma,t) = \overline{T}_0\, \overline{f}(\eta,\varsigma,t) \tag{24}$$

$$\overline{T}_n^* = f_n^* \ , \ \xi = b$$

Where $\overline{T}$ denotes the Marchi-Fasulo transform of T and ς denotes the Marchi-Fasulo transform parameter.

Applying Laplace transform defined in [5] to the equation (23) we get

$$\left[\frac{\partial^2 \overline{T}^*}{\partial \xi^2} + \frac{\partial^2 \overline{T}^*}{\partial \eta^2}\right] = 2q(\cosh 2\xi - \cos 2\eta)\overline{T}^* \tag{25}$$

Where

$$q=\left(\frac{ka_m^2+s}{k}\right)\frac{c^2}{4d^2}, \quad (26)$$

$\overline{T}^*$ is the Laplace transform of $\overline{T}$ and s is the Laplace transform parameter.
If the temperature is symmetric about both axes of elliptical plate, the appropriate solution of (23) is

$$\overline{T}^*(\xi,\eta,\varsigma,s)=\sum C_{2n}\left(f_n^*-\overline{T}_n^*\right)Ce_{2n}(\xi,-q)\times ce_{2n}(\eta,-q) \quad (27)$$

Where $Ce_{2n}(\xi,-q)$ and $ce_{2n}(\eta,-q)$ are defined as modified and ordinary Mathieu function of order 2n. Using (24) and (27) we get

$$\overline{f}^*(\xi,\eta,\varsigma,s)=\sum C_{2n}\left(f_n^*-\overline{T}_n^*\right)Ce_{2n}(\xi,-q)\times ce_{2n}(\eta,-q) \quad (28)$$

$$\overline{T}_n^*=\int_0^{2\pi}\overline{T}ce_{2n}(\eta,-q)d\eta \quad (29)$$

$$f_n^*=\int_0^{2\pi}\overline{f}(\eta)ce_{2n}(\eta,-q)d\eta \quad (30)$$

In order to get the value of constant C_{2n}, multiply (27) by $Ce_{2n}(\eta,-q)$ integrate with respect to η from 0 to 2π and making use of the following result:

$$\int_0^{2\pi}Ce_{2n}^2(\eta,-q)\,d\eta=\pi \quad \text{(Orthogonal properties)}$$

$$\int_0^{2\pi}Ce_{2n}(\eta,-q)\,\overline{f}(\eta,\varsigma)\,d\eta=\overline{f}_{2n}(\varsigma), \text{ (Say)} \quad (31)$$

we get

$$C_{2n}=\frac{1}{\pi}\sum_{n=0}^{\infty}\frac{\overline{f}_{2n}(\xi)}{Ce_{2n}(b,-q)} \quad (32)$$

Substitute (32) in equation (27) one obtains

$$\overline{T}^*(\xi,\eta,\varsigma,t)=\frac{1}{\pi}\sum_{n=0}^{\infty}\frac{\overline{f}_{2n}(\varsigma)\left(f_n^*-\overline{T}_n^*\right)}{Ce_{2n}(b,-q)}Ce_{2n}(\xi,-q)\times ce_{2n}(\eta,-q) \quad (33)$$

Further applying inverse Laplace transform and inverse Marchi-Fasulo transform to the equation (33) and using condition (22), one obtains the temperature distribution and unknown temperature gradient as

$$T(\xi,\eta,z,t)=\frac{1}{\pi}\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\frac{P_m(z)}{\lambda_m}\frac{Ce_{2n}(\xi,-q_{2n,m})\times ce_{2n}(\eta,-q_{2n,m})}{\frac{d}{dp}Ce_{2n}(b,-q_{2n,m})}$$
$$\times\int_0^t\overline{f}_{2n}(\xi)\left(f_n^*-\overline{T}_n^*\right)\exp(-k\lambda^2{}_{2n,m})(t-T)dT \quad (34)$$

$$g(\eta,z,t)=\frac{1}{\pi}\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\frac{P_m(z)}{\lambda_m}\frac{Ce_{2n}(\xi,-q_{2n,m})\times ce_{2n}(\eta,-q_{2n,m})}{\frac{d}{dp}Ce_{2n}(b,-q_{2n,m})}$$
$$\times\int_0^t\overline{f}_{2n}(\xi)\left(f_n^*-\overline{T}_n^*\right)\exp(-k\lambda^2{}_{2n,m})(t-T)dT \quad (35)$$

Where

$$q_{2n,m}=\left(ph^2+k\varsigma^2\pi^2\right)\frac{d^2}{4kh^2},$$

$$\overline{f}_{2n}(\xi,\eta,z)=\int_{-h}^{h}f(\xi,\eta,z,t)\,P_m(z)\,dz \;,$$

$$\lambda_m=\int_{-h}^{h}P_m^2(z)\,dz \;,$$

$$P_m(z)=Q_m\cos(a_mz)-W_m\sin(a_mz)$$

$$Q_m=a_m(\alpha_1+\alpha_2)\cos(a_mh)+(\beta_1-\beta_2)\sin(a_mh)$$

$$W_m=(\beta_1+\beta_2)\cos(a_mh)+(\alpha_2-\alpha_1)\sin(a_mh)$$

Equation (34) and (35) are the desired solution of the problem with $\beta_1=\beta_2=1$ and $\alpha_1=k_1$, $\alpha_2=k_2$.

6. CONCLUSION

In both the problems, we have determined the temperature distribution and unknown temperature gradient on outer curved surface of an elliptical plate with aid of Mathieu function and integral transform technique, using known boundary conditions. The results are obtained in the form of infinite series. The temperature distribution and unknown temperature gradient that are obtained can be applied to the design of useful structures or machines in engineering applications.

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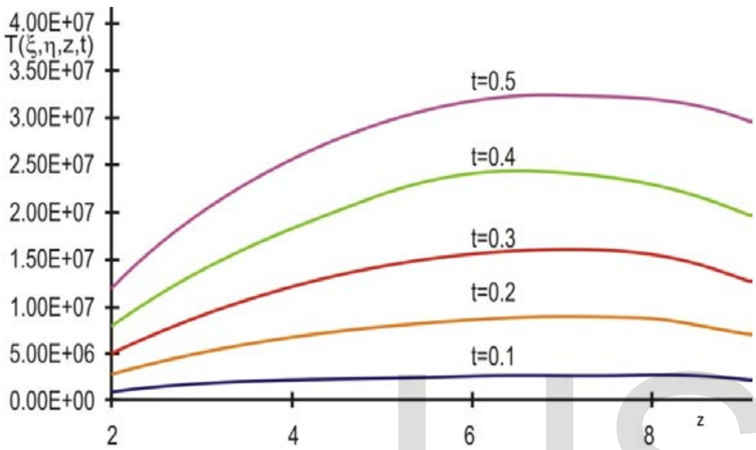
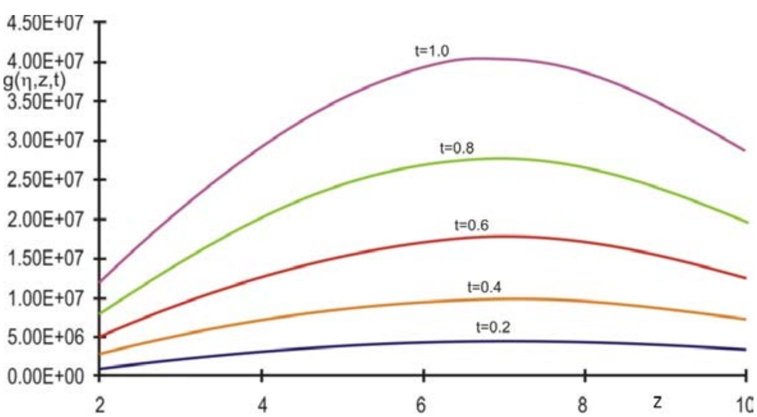


The Graph of $T(\xi,\eta,z,t)$ **versus z for different value of t.**



The Graph $g(\eta,z,t)$ Versus **z** for different value of t